

From Euclidean Spaces to Metric Spaces

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In this note, we provide the definition of a metric space and establish that, while all Euclidean spaces are metric spaces, not all metric spaces are Euclidean spaces. It is then natural and interesting to ask which theorems that hold in Euclidean spaces can be extended to general metric spaces and which ones cannot be extended. We survey this topic by considering six well-known theorems which hold in Euclidean spaces and rigorously exploring their validities in general metric spaces.

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Definition: Let M be a set. A metric on M is a function $d: M \times M \rightarrow \mathbb{R}$ such that for all $p, q, r \in M$, the following properties hold:

1. $d(p, q) \geq 0$;
2. $d(p, q) = d(q, p)$;
3. $d(p, q) = 0$ if and only if $p = q$;
4. $d(p, r) \leq d(p, q) + d(q, r)$.

The pair (M, d) is called a metric space.

We may deduce from this definition that each n -dimensional Euclidean space \mathbb{R}^n , where n is a positive integer, is a metric space when we define d as the known distance function (namely, for any points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in \mathbb{R}^n , $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$).

Definition: Let d and d' be two metrics on a set M . The metrics d and d' are called equivalent if there exist positive constants α and β such that $\alpha d'(x, y) \leq d(x, y) \leq \beta d'(x, y)$ for all $x, y \in M$. (This concept will prove useful shortly.)

On the other hand, not every metric space is a Euclidean space. For example, consider the set of real numbers \mathbb{R} and let d be the Euclidean metric on \mathbb{R} , that is, $d(x, y) = |x - y|$ for all x and y in \mathbb{R} . Let $d': \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $d'(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ for all $x, y \in \mathbb{R}$, where \mathbb{R} is the set of real numbers. (We reserve the symbol \mathbb{R} for the set of real numbers when this set is paired, either explicitly or implicitly, with the metric d .) It is clear that the first three properties of a metric listed above hold. To check the fourth property, there are three cases to be considered. First, assume that $x = y = z \in \mathbb{R}$. Then $d'(x, z) = d'(x, y) = d'(y, z) = 0$, and we see that $d'(x, z) \leq d'(x, y) + d'(y, z)$. Next, we assume that exactly two of these elements are equal, say (without loss of generality), $x = y \neq z$. In this case, $d'(x, y) = 0$ while $d'(x, z) = d'(y, z) = 1$ and so $1 = d'(x, z) \leq d'(x, y) + d'(y, z) = 1$. Finally, let x, y , and z be distinct. Then $d'(x, z) = 1$ while $d'(x, y) + d'(y, z) = 1 + 1 = 2$, and so $d'(x, z) \leq d'(x, y) + d'(y, z)$. It follows that the fourth listed property of a metric holds and (\mathbb{R}, d') is a metric space. Next we show that the metrics d and d' are not equivalent. On one hand, $d'(x, y) \leq 1$ for every x and y in \mathbb{R} ; on the other hand, $d(x, x + n) = n$ for any real number x and any positive integer n . Thus d is not bounded. It then follows that d' and d are not equivalent and so (\mathbb{R}, d') is not a Euclidean space. The fact that all Euclidean spaces are metric spaces while not all metric spaces are Euclidean spaces leads us to ask which theorems that hold in Euclidean spaces also hold in general metric spaces.

Convergence of Sequences

The first concept we explore is convergence of sequences in general metric spaces.

Definition: Suppose (E, d) is a metric space. Let $p_1, p_2, p_3, p_4, \dots$ be a sequence of points in E . We call a point p in E a limit of this sequence if, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow d(p, p_n) < \varepsilon$. If this is the case, the sequence is called convergent and is said to converge to p . (Otherwise the sequence is called divergent.)

Theorem: In the one-dimensional Euclidean space \mathbb{R} , every sequence has at most one limit.

Proof. Let $\{a_n\}$ be a sequence whose terms are in \mathbb{R} . If $\{a_n\}$ is divergent, then the sequence has no limits by definition and so the result follows immediately. Assume, on the other hand, that $\{a_n\}$ is convergent. Suppose, contrary to the result, that the sequence has distinct limits A and B in \mathbb{R} . Then for every $\varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow |a_n - A| < \varepsilon$ and $n \geq N_2 \Rightarrow |a_n - B| < \varepsilon$. Moreover, $|A - B| > 0$ because $A \neq B$. Let $n \geq \max\{N_1, N_2\}$. By the triangle inequality, we may then observe that $|A - B| \leq |A - a_n| + |a_n - B| < \varepsilon + \varepsilon = 2\varepsilon$, i.e., $|A - B| < 2\varepsilon$. Since this is true for every $\varepsilon > 0$, choose $\varepsilon = \frac{1}{2}|A - B| > 0$. Then $|A - B| < 2(\frac{1}{2}|A - B|) = |A - B|$, which is a contradiction. Hence our assumption that A and B are distinct must be false, so we conclude that $\{a_n\}$ converges to a unique limit. \square

We now prove that this result is true not only in Euclidean spaces, but in all metric spaces. The key in idea used in the proof is that for any two distinct points p_1 and p_2 in a metric space (E, d) , there exists $\varepsilon > 0$ such that $B(p_1, \varepsilon) \cap B(p_2, \varepsilon) = \emptyset$, where $B(p, \varepsilon) = \{x \in \mathbb{R} \mid d(p, x) < \varepsilon\}$ is the ε -ball centered at p in (E, d) .

Result 1: A sequence of points in a metric space has at most one limit in the metric space.

Proof. Suppose (E, d) is a metric space. Let $p_1, p_2, p_3, p_4, \dots$ be a sequence of points in E with metric $d: E \times E \rightarrow \mathbb{R}$. If the sequence is divergent, then it has no limits and so the result immediately follows. Assume, then, that the sequence is convergent. Suppose, contrary to the result, that the sequence has distinct limits p and q in E . Note that, because p and q are distinct, $d(p, q) > 0$. Then for every $\varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow d(p, p_n) < \varepsilon$ and $n \geq N_2 \Rightarrow d(q, p_n) < \varepsilon$. Let $n \geq \max\{N_1, N_2\}$. By the aforementioned fourth property of a metric, we see that $d(p, q) \leq d(p, p_n) + d(p_n, q) < \varepsilon + \varepsilon = 2\varepsilon$, i.e., $d(p, q) < 2\varepsilon$. Since this is true for every $\varepsilon > 0$, we choose $\varepsilon = \frac{1}{2}d(p, q) > 0$. Then $d(p, q) < 2(\frac{1}{2}d(p, q)) = d(p, q)$, which is a contradiction. Hence our assumption that p and q are distinct

must be false, so we conclude that the sequence $p_1, p_2, p_3, p_4, \dots$ converges to a unique limit. \square

Now that we have established that a sequence converges to at most one limit, this result may be incorporated (possibly in an implicit manner) into some of the results and proofs that follow. However, whether a given sequence is convergent is not the only curiosity we might have regarding said sequence: we might ask ourselves whether the sequence is Cauchy.

Definition: A sequence of points $p_1, p_2, p_3, p_4, \dots$ in a metric space (E, d) is Cauchy if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $n, m \geq N \Rightarrow d(p_n, p_m) < \varepsilon$.

Theorem: In \mathbb{R} , every convergent sequence is a Cauchy sequence.

Proof. Suppose the sequence $\{a_n\}$, whose terms are in \mathbb{R} , is convergent and that its (unique!) limit is A , which is also in \mathbb{R} . This means that $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - A| < \frac{\varepsilon}{2}$. (This is a valid interpretation of the definition of convergence, since $\varepsilon > 0 \Rightarrow \frac{\varepsilon}{2} > 0$.) Let $n, m \geq N$. By the triangle inequality, $|a_n - a_m| \leq |a_n - A| + |A - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This implies that $|a_n - a_m| < \varepsilon$, and so $\{a_n\}$ is Cauchy by definition. \square

An analogous proof shows that this too is a result that can be extended from Euclidean spaces to general metric spaces. Once again, the fourth property of a metric in our definition of a metric space proves to be valuable.

Result 2: In a metric space, a convergent sequence is a Cauchy sequence.

Proof. Let (E, d) be a metric space. Assume that the sequence of points $p_1, p_2, p_3, p_4, \dots$ in E with metric $d: E \times E \rightarrow \mathbb{R}$ is convergent, and suppose that its (again, unique!) limit is $p \in E$. Then $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow d(p_n, p) < \frac{\varepsilon}{2}$. Let $n, m \geq N$. We deduce from the fourth listed property of a metric that $d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, i.e., $d(p_n, p_m) < \varepsilon$. Hence, by definition, the sequence $p_1, p_2, p_3, p_4, \dots$ is Cauchy. \square

Continuity and Convergence

In order to set the stage for our next result, we now introduce the concept of continuity of a function that maps one metric space onto another.

Definition: Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f: (X, d_X) \rightarrow (Y, d_Y)$ be a function. We say that f is continuous at a point p in X provided that $\forall \varepsilon > 0, \exists \delta > 0$ such that $d_X(p, x) < \delta \Rightarrow d_Y(f(p), f(x)) < \varepsilon$. The function f is said to be a continuous function if this is true for each point of X .

Theorem: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous if and only if for every convergent sequence $x_n \rightarrow x$ in $\mathbb{R}, f(x_n) \rightarrow f(x)$.

Proof. (\Rightarrow) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Given $x_n \rightarrow x$, we wish to show that $f(x_n) \rightarrow f(x)$, i.e., for any $\varepsilon > 0$, there exists some positive integer N such that $|f(x_n) - f(x)| < \varepsilon$ for all $n \geq N$. Let us fix $\varepsilon > 0$. Since f is a continuous function, it follows from the definition that there exists $\delta > 0$ such that $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$ (where $x', f(x') \in \mathbb{R}$). Because $x_n \rightarrow x$, it follows from the definition of convergence that there exists $N \in \mathbb{N}$ such that $|x_n - x| < \delta$ for all $n \geq N$, which further implies that $|f(x_n) - f(x)| < \varepsilon$ for all $n \geq N$. Thus $f(x_n) \rightarrow f(x)$.

(\Leftarrow) Conversely, assume that for every sequence $a_n \rightarrow a$ in $\mathbb{R}, f(a_n) \rightarrow f(a)$. Our goal is to show that f is continuous. Suppose, to the contrary, that f is not continuous at x' in \mathbb{R} . Then $\exists \varepsilon > 0$

such that $\forall n \in \mathbb{N}, \exists x_n \in \mathbb{R}$ such that $|x_n - x'| < \frac{1}{n}$ and $|f(x_n) - f(x)| \geq \varepsilon$. This is a contradiction to our assumption because it means $x_n \rightarrow x$ while $f(x_n)$ does not converge to $f(x)$. Hence f must be continuous at x' in \mathbb{R} , and so f is a continuous function. \square

This is yet another result that is true in both Euclidean spaces and general metric spaces. We provide a proof of the result in general metric spaces by imitating our proof in the one-dimensional Euclidean space \mathbb{R} .

Result 3: Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f: (X, d_X) \rightarrow (Y, d_Y)$ be a function. Then f is continuous if and only if for every convergent sequence $x_n \rightarrow x$ in $X, f(x_n) \rightarrow f(x)$ in Y .

Proof. (\Rightarrow) Let $f: (X, d_X) \rightarrow (Y, d_Y)$ be a continuous function. Given $x_n \rightarrow x$ in X , we wish to show that $f(x_n) \rightarrow f(x)$ in Y , i.e., for any $\varepsilon > 0$, there exists some positive integer N such that $d_Y(f(x_n), f(x)) < \varepsilon$ for all $n \geq N$. Let us fix $\varepsilon > 0$. Since f is a continuous function, it follows from the definition there exists $\delta > 0$ such that $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$ (where $x' \in X, f(x') \in Y$). Because $x_n \rightarrow x$, it follows from the definition of convergence that there exists a positive integer N such that $d_X(x_n, x) < \delta$ for all $n \geq N$. This further implies that $d_Y(f(x_n), f(x)) < \varepsilon$ for all $n \geq N$. Thus $f(x_n) \rightarrow f(x)$ in Y .

(\Leftarrow) Conversely, assume that for every sequence $a_n \rightarrow a$ in $X, f(a_n) \rightarrow f(a)$ in Y . Our goal is to show that f is continuous. Suppose, to the contrary, that f is not continuous at x' in X . Then $\exists \varepsilon > 0$ such that $\forall n \in \mathbb{N}, \exists x_n \in X$ such that $d_X(x_n, x') < \frac{1}{n}$ and $d_Y(f(x_n), f(x')) \geq \varepsilon$. This is a contradiction to our assumption because it means $x_n \rightarrow x$ in X while $f(x_n)$ does not converge to $f(x)$ in Y . Hence f must be continuous at x' in X , and so f is a continuous function. \square

Completeness

Earlier we proved that in all metric spaces, every convergent sequence is Cauchy. In the one-dimensional Euclidean space \mathbb{R} , the converse is also true.

Theorem: In \mathbb{R} , every Cauchy sequence is convergent.

(Note: Our proof relies on the facts that every Cauchy sequence is bounded and that, according to the Bolzano-Weierstrass Theorem, every bounded sequence $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$. We do not prove these theorems because we wish to keep our focus on Euclidean and metric spaces, but they are helpful nonetheless.)

Proof. Suppose $\{a_n\}$, a sequence whose terms are in \mathbb{R} , is Cauchy. Then $\{a_n\}$ is bounded and, by the Bolzano-Weierstrass Theorem, has a subsequence $\{a_{n_k}\}$ that converges to A for some (unique!) real number A . Thus $\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}$ such that $k \geq N_1 \Rightarrow |a_{n_k} - A| < \frac{\varepsilon}{2}$. Moreover, that $\{a_n\}$ is Cauchy means that $\forall \varepsilon > 0, \exists N_2 \in \mathbb{N}$ such that $n, m \geq N_2 \Rightarrow |a_n - a_m| < \frac{\varepsilon}{2}$. Choose $N = \max\{N_1, N_2\}$. Since $n_N > N$, we have that $n \geq N \Rightarrow |a_n - A| = |(a_n - a_{n_N}) + (a_{n_N} - A)| \leq |a_n - a_{n_N}| + |a_{n_N} - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, i.e., $|a_n - A| < \varepsilon$. Thus, by definition, a_n converges to A in \mathbb{R} . \square

This shows that in \mathbb{R} , a sequence is convergent if and only if it is Cauchy, i.e., “convergent” and “Cauchy” are equivalent descriptions of a sequence in \mathbb{R} .

Definition: If every Cauchy sequence of points of a metric space is convergent to a point in the metric space, then the metric space is called complete.

We just showed that \mathbb{R} is complete, but not all metric spaces share that title. Before proving this, let us first establish a lemma.

Lemma: If (M, d) is a metric space, $E \subset M$, and $d': E \times E \rightarrow \mathbb{R}$ is the restriction of d on $E \times E$, then (E, d') is a metric space.

Proof. Let (M, d) be a metric space, and assume that $E \subset M$. Then for every $x, y, z \in E \subset M$, since d is a metric on M and d' is the restriction of d on $E \times E$, it follows that

- (1) $d(x, y) = d'(x, y) \geq 0$;
- (2) $d'(x, y) = d(x, y) = d(y, x) = d'(y, x)$;
- (3) $d(x, y) = d'(x, y) = 0$ if and only if $x = y$;
- (4) $d'(x, z) = d(x, z) \leq d(x, y) + d(y, z) = d'(x, y) + d'(y, z)$.

Hence it follows immediately from the definition that (E, d') is a metric space. \square

Result 4: In general metric spaces, not every Cauchy sequence is necessarily a convergent sequence.

Proof. Consider the sequence $\{a_n\} = \{\frac{1}{n} : n \in \mathbb{N}\}$ in the one-dimensional Euclidean space \mathbb{R} . We first show that $a_n \rightarrow 0$ in \mathbb{R} . Let $\varepsilon > 0$. Then $|a_n - 0| < \varepsilon \Leftrightarrow |a_n| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$. Choose $N = \lceil \frac{1}{\varepsilon} \rceil + 1 \in \mathbb{N}$. Then $\forall \varepsilon > 0, \exists N = \lceil \frac{1}{\varepsilon} \rceil + 1 \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - 0| < \varepsilon$. Hence, by definition, $a_n \rightarrow 0$ in \mathbb{R} . Since $\{a_n\}$ is convergent in \mathbb{R} , it follows from Result 2 that $\{a_n\}$ is Cauchy in \mathbb{R} . This means that $\forall \varepsilon > 0, \exists N' \in \mathbb{N}$ such that $n, m \geq N' \Rightarrow |a_n - a_m| < \varepsilon$. Suppose, now, that the sequence is not in \mathbb{R} but rather in $\mathbb{R} - \{0\}$ (with the Euclidean metric still applied). By the lemma preceding this result, the set $\mathbb{R} - \{0\}$ paired with the one-dimensional Euclidean metric is indeed a metric space. Observe that the sequence is Cauchy in $\mathbb{R} - \{0\}$ by definition (after all, 0 is not even a term of the sequence and so its removal from the original set does not alter this conclusion), but its (unique!) limit 0 is clearly not in $\mathbb{R} - \{0\}$. Therefore, $\{a_n\}$ is Cauchy but not convergent in $\mathbb{R} - \{0\}$. \square

The Intermediate Value Theorem

Thus far, much of the material we have discussed has been accessible primarily to a mathematically mature audience. However, we will now explore a theorem that should look familiar to any diligent student of introductory calculus: the Intermediate Value Theorem.

Theorem (Intermediate Value Theorem): If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function that is continuous on the interval $[a, b]$ in \mathbb{R} and r is a real number strictly between $f(a)$ and $f(b)$, then there is some real number c in the interval (a, b) such that $f(c) = r$.

(Note: We wish to avoid being too pedantic, so let us take for granted this fact: if f is continuous on some interval J of real numbers and x and y are in J with $x < y$, then $f(x)f(y) < 0 \Rightarrow \exists c \in (a, b)$ such that $f(c) = 0$. To state this intuitively, a function $f: \mathbb{R} \rightarrow \mathbb{R}$ that is continuous on an interval whose endpoints lie on opposite sides of the x -axis must cross the x -axis at some point in the interval. With this in mind, we prove the Intermediate Value Theorem for functions mapping the one-dimensional Euclidean space \mathbb{R} onto itself.)

Proof. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function that is continuous on the interval $[a, b]$ in \mathbb{R} . Assume that r is a real number strictly between $f(a)$ and $f(b)$, and note that $k(x) = f(x) - r$ is continuous on $[a, b]$. Since $k(a)k(b) = (f(a) - r)(f(b) - r) < 0$, we know by the above fact that $\exists c \in (a, b)$ such that $k(c) = f(c) - r = 0$, i.e., $f(c) = r$. \square

While the Intermediate Value Theorem may seem instinctive when working in Euclidean spaces, it does not hold in all metric spaces. We prove this by considering the domain of a function to be the union of two disjoint intervals of real numbers.

Result 5: The Intermediate Value Theorem does not hold in general metric spaces.

Proof. Let (X, d) be a metric space, and assume that d is the one-dimensional Euclidean metric (i.e., for every x and y in X , $d(x, y) = |x - y|$). Suppose $X = (-\infty, -1) \cup (1, \infty) \subset \mathbb{R}$. Let $f: (X, d) \rightarrow \mathbb{R}$ be a function such that $f(x) = \begin{cases} -1, & x \in (-\infty, -1] \\ 1, & x \in [1, \infty) \end{cases}$, and observe that f is a continuous function because it is continuous at each point in its domain. Clearly $f(-2) = -1$ and $f(2) = 1$, but there is no c in the domain of f such that $f(c) = 0$ even though $-1 < 0 < 1$. Hence the Intermediate Value Theorem does not hold in this metric space. \square

Compactness

The final major idea we explore is compactness of metric spaces, and we begin with terminology that will help us describe this concept.

Definition: Let (X, d) be a metric space. A cover of X is a collection of sets whose union is exactly the set X . If there is a collection of open sets whose union is X , we call this collection an open cover. We say that the metric space is compact if every cover of X has a finite subcover.

Theorem: In each n -dimensional Euclidean space \mathbb{R}^n , every closed bounded subset is compact. We treat this as a fact and omit the proof in order to stay within the scope of this work, but we will now show that this statement is not true in general metric spaces.

Result 6: In general metric spaces, a closed bounded subset is not necessarily compact.

Proof. Consider the metric space (R, d) , where R denotes the set of real numbers and $d: R \times R \rightarrow \mathbb{R}$ is a function such that $d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ for all $x, y \in R$. (We showed in the introduction that this is indeed a valid metric.) In the metric space (R, d) , for any subset M , M is bounded in (R, d) . Moreover, R is a closed subset of itself. It follows that R is a closed bounded subset of (R, d) . Consider the open cover of R , $\{B(x, \frac{1}{2}) \mid x \in R\}$, where $B(x, \frac{1}{2}) = \{y \in R \mid d(x, y) < \frac{1}{2}\}$. By the definition of the metric d , $B(x, \frac{1}{2}) = \{x\}$. It is clear that the open cover $\{\{x\} \mid x \in R\}$ has no finite subcover, and so, in the metric space (R, d) , the closed bounded subset R is not compact. \square

Conclusion

Throughout this paper, we studied Euclidean spaces and metric spaces through the lenses of convergence, completeness, continuity, and compactness. As one may have anticipated, we discovered that a number of the results we examined could be extended from Euclidean spaces to general metric spaces while others could not. The ability to define a metric as any "distance" function over a set satisfying just the four properties listed in the definition of a metric space leads to a sharp decrease in one's ability to rely on intuition when working in general metric spaces as opposed to the Euclidean spaces we all know and love. Our work largely serves as motivation to both appreciate concrete mathematical structures

and consider how the properties of these structures change when we abstract them.

Reference

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